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Note

On new generalizations of the Hilbert integral inequality

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Abstract

Some new generalizations of the Hilbert integral inequality by introducing real functions $\phi(x)$ and $\psi(x)$. The results of this paper reduce to those of the corresponding inequalities proved by Gao [Mingzhe Gao, On Hilbert's integral inequality, Math. Appl. 11 (3) (1998) 32–35]. Some applications are considered.

Keywords: Inner product; Hilbert integral inequality; Hardy–Littlewood integral inequality

1. Introduction

Let $f(x), g(x) \in L^2(0, \infty)$. Then Hilbert's integral inequality may be written as

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy\right)^2 \leqslant \pi^2 \int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy, \tag{1}$$

where π is the best value started by Hardy et al. [1, Chapter 9].

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Recently, Gao [2] gave an improvement of (1) by means of Gram's matrix as follows:

$$\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy\right)^2 \leqslant \pi^2\int_0^\infty f^2(x)\,dx\int_0^\infty g^2(y)\,dy - F(s,t), \tag{2}$$

where the function F is defined by

$$F(s,t) = \|\alpha\|^2 s^2 - 2(\alpha,\beta)st + \|\beta\|^2 t^2, \tag{3}$$

where α, β and γ are three arbitrary vectors of inner product space E, (α,β) indicates the inner product of the vectors α and β, and γ is a unit-vector, the α and β are not simultaneously orthogonal to the vector γ, and $s = (\beta,\gamma)$ and $t = (\alpha,\gamma)$ are the functionals of the vector γ of E.

Based on the method of Gao [2], we further make some new generalizations of Hilbert integral inequality by introducing real functions $\phi(x)$ and $\psi(x)$. Some Hilbert type integral inequalities are obtained. Some applications are also considered.

2. Main results

First we give some lemmas.

Lemma 2.1. (See Gao [2].) *Let $F(s,t)$ be the function defined by* (3). *If the vectors α and β are linearly independent, then $F(s,t) > 0$ for any $s,t \in \mathbb{R}$, provided that s and t are not zero simultaneously.*

Lemma 2.2. (See Gao [2].) *Let α and β be two given vectors of E, and γ be an arbitrary vector of E. If the vectors α and β are linearly independent, and $\|\gamma\| = 1$, then*

$$(\alpha,\beta)^2 \leqslant \|\alpha\|^2\|\beta\|^2 - F(s,t), \tag{4}$$

where $F(s,t)$ is the function defined by (3) *and $F(s,t) > 0$. And the equality contained in* (4) *holds if and only if the vector γ is a linear combination of the vectors α and β.*

Theorem 2.1. *Let $f, g \in L^2(0,\infty)$ and $\phi(x)$ and $\psi(x)$ be differentiable functions in* $(0,+\infty)$, *and $\phi(0), \psi(0) \geqslant 0$, $\phi(\infty) = \infty$, $\psi(\infty) = \infty$, $\phi(0) = \psi(0)$, $\phi'(x), \psi'(x) > 0$, $\phi'(x)$ and $\psi'(x)$ have positive infimum, respectively, then*

$$\begin{aligned}\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\phi(x)+\psi(y)}\,dx\,dy\right)^2 &\leqslant \frac{1}{\inf\{\phi'(x)\}\inf\{\psi'(y)\}}\int_0^\infty\left(\pi - \arctan\sqrt{\frac{\psi(0)}{\phi(x)}}\right)f^2(x)\,dx \\ &\quad\times\int_0^\infty\left(\pi - \arctan\sqrt{\frac{\phi(0)}{\psi(y)}}\right)g^2(y)\,dy - F(s,t),\end{aligned} \tag{5}$$

where $F(s,t)$ is the function defined by (3) *and $F(s,t) > 0$.*

Proof. We define two functions by

$$\alpha = \frac{f(x)}{(\phi(x)+\psi(y))^{1/2}}\left(\frac{\phi(x)}{\psi(y)}\right)^{1/4}, \qquad \beta = \frac{g(y)}{(\phi(x)+\psi(y))^{1/2}}\left(\frac{\psi(y)}{\phi(x)}\right)^{1/4}. \tag{6}$$

We may apply inequality (4) to estimate the left-hand side of (5) as follows:

$$\begin{aligned}
&\left(\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\phi(x)+\psi(y)}\,dx\,dy\right)^2 \\
&\quad= \left(\int_0^\infty\int_0^\infty \alpha\beta\,dx\,dy\right)^2 \\
&\quad\leqslant \int_0^\infty\int_0^\infty \alpha^2\,dx\,dy \int_0^\infty\int_0^\infty \beta^2\,dx\,dy - F(s,t) \\
&\quad= \int_0^\infty\left(\int_0^\infty \frac{1}{\phi(x)+\psi(y)}\left(\frac{\phi(x)}{\psi(y)}\right)^{1/2} dy\right) f^2(x)\,dx \\
&\qquad\times \int_0^\infty\left(\int_0^\infty \frac{1}{\phi(x)+\psi(y)}\left(\frac{\psi(y)}{\phi(x)}\right)^{1/2} dx\right) g^2(y)\,dy - F(s,t) \\
&\quad= \int_0^\infty \omega(\phi,\psi,x) f^2(x)\,dx \int_0^\infty \omega(\psi,\phi,y) g^2(y)\,dy - F(s,t),
\end{aligned} \tag{7}$$

where the weight function $\omega(\phi,\psi,x)$ is defined by

$$\omega(\phi,\psi,x) = \int_0^\infty \frac{1}{\phi(x)+\psi(y)}\left(\frac{\phi(x)}{\psi(y)}\right)^{1/2} dy.$$

Putting $u = \frac{\psi(y)}{\phi(x)}$, we have

$$\begin{aligned}
\omega(\phi,\psi,x) &\leqslant \frac{1}{\inf\{\psi'(y)\}} \int_{\frac{\psi(0)}{\phi(x)}}^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2} du \\
&= \frac{1}{\inf\{\psi'(y)\}}\left(\pi - 2\arctan\sqrt{\frac{\psi(0)}{\phi(x)}}\right).
\end{aligned} \tag{8}$$

Similarly we prove

$$\begin{aligned}
\omega(\psi,\phi,y) &\leqslant \frac{1}{\inf\{\phi'(x)\}} \int_{\frac{\phi(0)}{\psi(y)}}^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{1/2} du \\
&= \frac{1}{\inf\{\phi'(x)\}}\left(\pi - 2\arctan\sqrt{\frac{\phi(0)}{\psi(y)}}\right).
\end{aligned} \tag{9}$$

It follows from (7)–(9) that inequality (5) remains valid. Obviously, α and β defined by (6) are independent, by Lemma 2.1 we have $F(s,t) > 0$. And the theorem is proved. □

Corollary 2.1. *Let $f \in L^2(0,\infty)$ and $\phi(x)$ be a differentiable function in $(0,\infty)$. If $\phi(0) \leqslant 0$, $\phi(\infty) = \infty$, $\phi'(x) > 0$ and $\phi'(x)$ has a positive infimum in $(0,\infty)$, then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{\phi(x)+\phi(y)}\,dx\,dy\right)^2 \leqslant \frac{1}{\inf\{\phi'(x)\}^2}\left(\int_0^\infty \left(\pi - 2\arctan\sqrt{\frac{\phi(0)}{\phi(x)}}\right) f^2(x)\,dx\right)^2 - F(s,t), \tag{10}$$

where $F(s,t)$ is the function defined by (3) *and $F(s,t) > 0$.*

Remark 2.1. When $\phi(x) = \psi(x) = x$, inequalities (5) and (10) reduce to (5) and (7) of the paper by Gao [2], respectively, hence inequalities (5) and (10) in this paper are generalizations of the corresponding results of Gao [2].

3. Some applications

We take $\phi(x)$ and $\psi(x)$ as

$$\phi(x) = e^x, \qquad \psi(y) = e^y,$$

then, by Theorem 2.1, we get the following result:

Theorem 3.1. *Let $f, g \in L^2(0,\infty)$. Then*

$$\left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{e^x+e^y}\,dx\,dy\right)^2 \leqslant \int_0^\infty \left(\pi - 2\arctan\sqrt{\frac{1}{e^x}}\right) f^2(x)\,dx \times \int_0^\infty \left(\pi - 2\arctan\sqrt{\frac{1}{e^y}}\right) g^2(y)\,dy - F(s,t), \tag{11}$$

where $F(s,t)$ is defined by (3) *and $F(s,t) > 0$.*

Proof. We define two functions by

$$\alpha = \frac{f(x)}{(e^x+e^y)^{1/2}} e^{\frac{x-y}{4}}, \qquad \beta = \frac{g(y)}{(e^x+e^y)^{1/2}} e^{\frac{y-x}{4}}. \tag{12}$$

A proof similar to that of Theorem 2.1 can be used to obtain inequality (11). We omit the proof. □

Theorem 3.2. *Let $g \in L^2(0,1)$ with $g(t) \neq 0$ for all $t \in (0,1)$, $\phi(x)$ be a differentiable function in $(0,\infty)$, and $\phi(0) \geqslant 0$, $\phi(\infty) = \infty$, $\phi'(x) > 0$ and $\phi'(x)$ has a positive infimum, and define a function F by*

$$F(x) = \int_0^1 u^{\phi(x)} g(u)\, du \quad (x \geqslant 0),$$

then

$$\left(\int_0^\infty f^2(x)\, dx\right)^2 \leqslant \left\{\left(\frac{1}{(\inf\{\phi'(x)\})^2} \int_0^\infty \left(\pi - 2\arctan\sqrt{\frac{\phi(0)}{\phi(x)}}\right) f^2(x)\, dx\right)^2 - F_2(s,t)\right\}^{1/2} \times \int_0^1 u g^2(u)\, du - F_1(s,t), \tag{13}$$

where $F_1(s,t)$, $F_2(s,t)$ are defined by (3) *and $F_1(s,t) > 0$, $F_2(s,t) > 0$.*

Proof. We may write $f^2(x)$ in the form

$$f^2(x) = \int_0^1 f(x) u^{\phi(x)} g(u)\, du.$$

We next apply inequalities (4) and (5) successively to estimate the left-hand side of (13) as follows:

$$\begin{aligned}
\left(\int_0^\infty f^2(x)\, dx\right)^2
&= \left\{\int_0^\infty \left(\int_0^1 f(x) u^{\phi(x)} g(u)\, du\right) dx\right\}^2 \\
&= \left\{\int_0^1 \left(\int_0^\infty f(x) u^{\phi(x)-1/2}\, dx\right) \left(u^{1/2} g(u)\right) du\right\}^2 \\
&\leqslant \int_0^1 \left(\int_0^\infty f(x) u^{\phi(x)-1/2}\, dx\right)^2 du \int_0^1 u g^2(u)\, du - F_1(s,t) \\
&= \int_0^1 \left(\int_0^\infty f(x) u^{\phi(x)-1/2}\, dx\right) \left(\int_0^\infty f(y) u^{\phi(y)-1/2}\, dy\right) du \int_0^1 u g^2(u)\, du - F_1(s,t)
\end{aligned}$$

$$= \int_0^1 \left(\int_0^\infty \int_0^\infty f(x) f(y) u^{\phi(x)+\phi(y)-1}\, dx\, dy \right) du \int_0^1 u g^2(u)\, du - F_1(s,t)$$

$$= \int_0^\infty \int_0^\infty \frac{f(x) f(y)}{\phi(x)+\psi(y)}\, dx\, dy \int_0^1 u g^2(u)\, du - F_1(x,y)$$

$$\leqslant \left\{ \left(\frac{1}{(\inf\{\phi'(x)\})^2} \int_0^\infty \left(\pi - 2\arctan\sqrt{\frac{\phi(0)}{\phi(x)}} \right) f^2(x)\, dx \right)^2 - F_2(s,t) \right\}^{1/2}$$

$$\times \int_0^1 u\, g^2(u)\, du - F_1(s,t). \tag{14}$$

Since $g(u) \neq 0$ for all $u \in (0,1)$, we have $f(x) \neq 0$. Hence $F_1(s,t) > 0$ and $F_2(s,t) > 0$. Thus, theorem is proved. □

Remark 3.1. For $\phi(x) = \psi(x) = x$, inequality (13) reduces to inequality (13) proved by Gao [2]. Hence inequality (13) is a generalization of the corresponding result of Gao [2].

Remark 3.2. If $F_1(s,t)$ and $F_2(s,t)$ contained in (13) are replaced by zero simultaneously, then inequality (13) reduces to the following form:

$$\int_0^\infty f^2(x)\, dx < \frac{1}{(\inf\{\phi'(x)\})^2} \int_0^1 u g^2(u)\, du.$$

Clearly, this is also a generalization of the Hardy–Littlewood inequality (see Hardy et al. [1]).

References

[1] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, London, 1952.
[2] Mingzhe Gao, On Hilbert's integral inequality, Math. Appl. 11 (3) (1998) 32–35.